ON A GENERAL EQUATION OF MOTION

The Case of an Azimuthally Symmetric Gravitational Field

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ABSTRACT

We derive the empirical flyby anomaly formula of Anderson et al. (2008) from the Azimuthally Symmetric Theory of Gravitation (ASTG), showing that first-order ($\ell = 1$) orientation-dependent corrections to a $1/r$ potential produce the geometric dependence

$$\frac{\Delta v_\infty}{v_\infty} = \kappa_A(\cos\delta_{\rm in} - \cos\delta_{\rm out}).$$

From the first-order ASTG potential we obtain explicit non-central accelerations (radial and polar components), demonstrate an energy-balance relation connecting perigee and asymptotic anomalies, and relate the empirical constant κ_A to ASTG parameters. The derivation and the merged, minute-resolved ephemerides + EOP dataset for seven flybys together form a reproducible framework for testing ASTG against spacecraft data.

INTRODUCTION

Spacecraft Earth flybys have revealed small but puzzling velocity anomalies (the Flyby Anomaly, FBA) with unaccounted shifts at perigee and in outgoing asymptotic speed, all of which resist a fully satisfactory Newtonian/GR explanation.

The Azimuthally Symmetric Theory of Gravitation (ASTG) extends a $1/r$ potential to allow dependence on the polar angle θ, introducing orientation-dependent terms tied to the central body's spin.

We present the ASTG potential, derive the associated non-central force components, exhibit the conserved specific-energy relations relevant to flybys, and show how the Anderson empirical law naturally emerges from the ASTG first-order term.

THEORY: ASTG POTENTIAL

ASTG potential (general form)

$$\Phi(r,\theta) = -\frac{GM_g}{r}\left[1 + \sum_{\ell=1}^{\infty}\lambda_\ell\left(\frac{R_s}{2r}\right)^{\ell} P_\ell(\cos\theta)\right].$$

First-order ($\ell = 1$) approximation

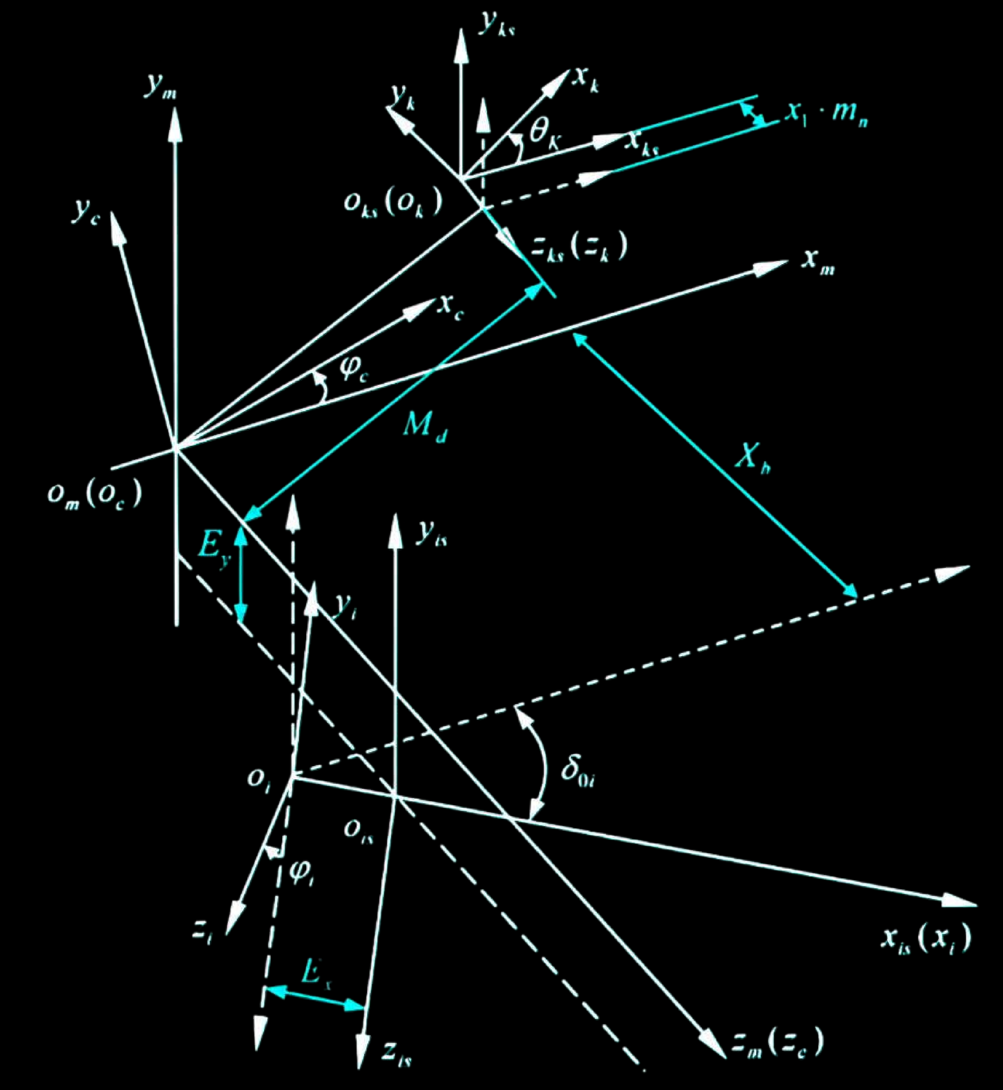
$$\Phi(r,\theta) \approx -\frac{GM_g}{r}\left(1 + \lambda_1\frac{R_s}{2r}\cos\theta\right).$$

ASTG acceleration spherical components

$$a_r = -\frac{GM}{r^2}\left[1 + \sum_{\ell=1}^{\infty}(\ell+1)\lambda_\ell\left(\frac{R_s}{r}\right)^{\ell} P_\ell(\cos\theta)\right],$$

$$a_\theta = -\frac{GM}{r^2}\sum_{\ell=1}^{\infty}\lambda_\ell\left(\frac{R_s}{r}\right)^{\ell} P_\ell^1(\cos\theta), \quad a_\phi = 0.$$

The non-zero a_θ is the non-central force responsible for deviations.



Azimuth angle (θ_k): the angle between the reference x-axis and the projection of the rotated axis x_k onto the horizontal plane (x–y), denoted x_{ks}.

DERIVATION SKETCH

Energy statement. Given that, G is Newton's constant, M_g is gravitational mass, R_s is characteristic radius, P_ℓ are Legendre polynomials and λ_ℓ are dimensionless ASTG parameters then specific energy (ASTG gravity only):

$\mathcal{E} = \frac{1}{2}v^2 + \Phi(r,\theta) = \frac{1}{2}v_\infty^2$ (incoming/outgoing $r \to \infty$).

Using the first-order ASTG potential yields expressions for $v^2(r,\theta)$ and differences between asymptotic states.

Energy-balance empirical relation

$$v_{\rm prg}\,\Delta v_{\rm prg} \approx v_\infty\,\Delta v_\infty,$$

plotted as $Y = v_{\rm prg}\Delta v_{\rm prg}$ vs $X = v_\infty\Delta v_\infty$. The data show near-perfect linearity (slope $\approx$ 1.001, $R^2 \approx$ 0.99999), supporting an energy-conserving anomalous force interpretation.

Connection to Anderson empirical law

$$\frac{\Delta v_\infty}{v_\infty} = \kappa_\oplus^A\big(\cos\delta_{\rm in} - \cos\delta_{\rm out}\big),$$

where $\kappa_\oplus^A = (3.10 \pm 0.06) \times 10^{-6}$. The first-order ASTG derivation reproduces this dependence and links $\kappa_\oplus^A$ to ASTG parameter combinations ($\kappa_\oplus^A = \lambda_1 R_s/\ell$).

WORK INTEGRAL

(A) Work integral & small-Δv_∞

Total specific energy (ASTG only):

$$\mathcal{E} = \tfrac{1}{2}v^2 + \Phi(r,\theta).$$

The change in asymptotic kinetic energy produced by the perturbing acceleration $\mathbf{a}_{\rm pert}$ is the net work along the unperturbed trajectory:

$$\Delta\left(\tfrac{1}{2}v^2\right) = \int_{-\infty}^{\infty}\mathbf{v}(t)\cdot\mathbf{a}_{\rm pert}(t)\,dt.$$

For small fractional changes ($\Delta v_\infty \ll v_\infty$):

ENERGY CHANGE

$$\Delta v_\infty \approx \frac{1}{v_\infty}\int_{-\infty}^{\infty}\mathbf{v}(t)\cdot\mathbf{a}_{\rm pert}(t)\,dt.$$

(B) Practical (Cowell) approximation

Replace $(\mathbf{v}(t), r(t), \theta(t))$ by unperturbed expressions. For $\ell = 1$ ASTG ($\propto \lambda_1\cos\theta$), the integral reduces to the Anderson geometric dependence:

$$\Delta v_\infty \propto v_\infty(\cos\delta_{\rm in} - \cos\delta_{\rm out}).$$

Explicit integral for numerical evaluation:

$$\Delta v_\infty \approx -\frac{GM\lambda_1\alpha}{v_\infty}\int_{-\infty}^{\infty}\frac{2\cos\theta v_r + \sin\theta v_\theta}{r^3}\,dt.$$

(C) Mapping to Anderson's constant κ_A

Anderson et al. wrote $\frac{\Delta v_\infty}{v_\infty} = \kappa_A\big(\cos\delta_{\rm in} - \cos\delta_{\rm out}\big)$. Numeric example:

$$\kappa_A = \frac{2R_\oplus\omega_\oplus}{c} \approx 3.0993 \times 10^{-6}.$$

Estimating λ_1 (back-of-envelope):

$$\lambda_1 \sim 2\kappa_A \approx 6.20 \times 10^{-6}.$$

KEY TAKEAWAYS

- The first-order ASTG correction produces explicit non-central accelerations that produce the Anderson geometric dependence.
- The merged flyby data show an empirical energy balance $v_{\rm prg}\Delta v_{\rm prg} \approx v_\infty\Delta v_\infty$.
- The empirical constant $\kappa_\oplus^A$ can be used to constrain ASTG parameter λ_1.

REFERENCES AND CONTACTS

[1] Anderson, J. D., Laing, P. A., Lau, E. L., Liu, A. S., Nieto, M. M. & Turyshev, S. G. (2002), 'Study of The Anomalous Acceleration of Pioneer 10 and 11', Physical Review D 65(8), 082004.

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